

Exploring super-radiant phase transitions via coherent control of a multi-qubit–cavity system

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Abstract. We propose the use of coherent control of a multi-qubit–cavity QED system in order to explore novel phase transition phenomena in a general class of multi-qubit–cavity systems. In addition to atomic systems, the associated super-radiant phase transitions should be observable in a variety of solid-state experimental systems, including the technologically important case of interacting quantum dots coupled to an optical cavity mode.

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1. Introduction

There is much current interest in the use of coherent control in order to generate novel matter-radiation states in cavity QED and atom-optics systems [1]. In addition, the field of cavity QED has caught the interest of workers in the field of solid-state nanostructures, since effective two-level systems can be fabricated using semiconductor quantum dots, organic molecules and even naturally-occurring biological systems such as the photosynthetic complexes LHI and LHII and in biological imaging setups involving FRET (Fluorescence Resonance Energy Transfer) [2]. Such nanostructure systems could then be embedded in optical cavities or their equivalent, such as in the gap of a photonic band-gap material [3]. We refer to Ref. [4] for a discussion of the size and energy-gaps of the artificial nanostructure systems which can currently be fabricated experimentally.

In a parallel development, phase transitions in quantum systems are currently attracting much attention within the solid-state, atomic and quantum information communities [5, 6, 7, 8]. Most of the focus within the solid-state community has been on phase transitions in electronic systems such as low-dimensional magnets [5, 6] while in atomic physics there has been much interest in phase transitions in cold atom gases and in atoms coupled to a cavity. In particular, a second-order phase transition, from

normal to superradiant, is known to arise in the Dicke model which considers N two-state atoms (i.e. ‘spins’ or ‘qubits’ [7, 8]) coupled to an electromagnetic field (i.e. bosonic cavity mode) [9, 10, 11]. The Dicke model itself has been studied within the atomic physics community for fifty years, but has recently caught the attention of solid-state physicists working on arrays of quantum dots, Josephson junctions, and magnetoplasmas [13]. Its extension to quantum chaos [14], quantum information [15] and other exactly solvable models has also been considered recently [16]. It has also been conjectured that superradiance could be used as a mechanism for quantum teleportation [17].

Here we extend our discussion in Ref. [18] on the exploration of novel phase transitions in atom-radiation systems exploiting the current levels of experimental expertise in the area of coherent control. The corresponding experimental set-up can be a cavity-QED, atom-optics, or nanostructure-optics system, whose energy gaps and interactions are tailored to be the required generalization of the well-known Dicke model [11]. We show that, according to the values of these control parameters, the phase transitions be driven to become first-order.

2. The Model

The well-known Dicke model from atom-optics ignores interactions between the constituent two-level systems or ‘spins’ [11]. In atomic systems where each ‘spin’ is an atom, this is arguably an acceptable approximation if the atoms are neutral *and* the atom-atom separation $d \gg a$ where a is the atomic diameter. However there are several reasons why this approximation is unlikely to be valid in typical solid-state systems. First, the ‘spin’ can be represented by any nanostructure (e.g. quantum dot) possessing two well-defined energy levels, yet such nanostructures are not typically neutral. Hence there will in general be a short-ranged (due to screening) electrostatic interaction between neighbouring nanostructures. Second, even if each nanostructure is neutral, the typical separation distance d between fabricated and self-organised nanostructures is typically the same as the size of the nanostructure itself. Hence neutral systems such as excitonic quantum dots will still have a significant interaction between nearest neighbors [19].

Motivated by the experimental relevance of ‘spin-spin’ interactions, we introduce and analyze a generalised Dicke Hamiltonian which is relevant to current experimental setups in both the solid-state and atomic communities [20]. We show that the presence of transverse spin-spin coupling terms, leads to novel first-order phase transitions associated with super-radiance in the bosonic cavity field. A technologically important example within the solid-state community would be an array of quantum dots coupled to an optical mode. This mode could arise from an optical cavity, or a defect mode in a photonic band gap material [20]. However we emphasise that the N ‘spins’ may correspond to *any* two-level system, including superconducting qubits and atoms [13, 20]. The bosonic field is then any field to which the corresponding spins couple [13, 20]. Apart from the experimental prediction of novel phase transitions, our work also provides an

interesting generalisation of the well-known Dicke model.

The method of solution that we present here is in fact valid for a wider class of Hamiltonians incorporating spin-spin and spin-boson interactions [21]. We follow the method of Wang and Hioe [11], whose results also proved to be valid for a wider class of Dicke Hamiltonians. We focus on the simple example of the Dicke Hamiltonian with an additional spin-spin interaction in the y direction.

$$H = a^\dagger a + \sum_{j=1}^N \left\{ \frac{\lambda}{2\sqrt{N}} (a + a^\dagger) (\sigma_j^+ + \sigma_j^-) + \frac{\epsilon}{2} \sigma_j^Z - J \sigma_j^Y \cdot \sigma_{j+1}^Y \right\} \quad (1)$$

$$= a^\dagger a + \sum_{j=1}^N \left\{ \frac{\lambda}{\sqrt{N}} (a + a^\dagger) \sigma_j^X + \frac{\epsilon}{2} \sigma_j^Z - J \sigma_j^Y \cdot \sigma_{j+1}^Y \right\}. \quad (2)$$

Following the discussion above, the experimental spin-spin interactions are likely to be short-ranged and hence only nearest-neighbor interactions are included in H . The operators in Eqs. 1 and 2 have their usual, standard meanings.

3. Results

To obtain the thermodynamical properties of the system, we first introduce the Glauber coherent states $|\alpha\rangle$ of the field [12] where $a|\alpha\rangle = \alpha|\alpha\rangle$, $\langle\alpha|a^\dagger = \langle\alpha|\alpha^*$. The coherent states are complete, $\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1$. In this basis, we may write the canonical partition function as:

$$Z(N, T) = \sum_{\mathbf{s}} \int \frac{d^2\alpha}{\pi} \langle \mathbf{s} | \langle \alpha | e^{-\beta H} | \alpha \rangle | \mathbf{s} \rangle \quad (3)$$

As in Ref. [11], we adopt the following assumptions:

- (i) a/\sqrt{N} and a^\dagger/\sqrt{N} exist as $N \rightarrow \infty$;
- (ii) $\lim_{N \rightarrow \infty} \lim_{R \rightarrow \infty} \sum_{r=0}^R \frac{(-\beta H_N)^r}{r!}$ can be interchanged

We then find

$$Z(N, T) = \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} \text{Tr} e^{-\beta H'} \quad (4)$$

where

$$H' = \sum_{j=1}^N \left\{ \frac{2\lambda \text{Re}(\alpha)}{\sqrt{N}} \sigma_j^X + \frac{\epsilon}{2} \sigma_j^Z - J \sigma_j^Y \cdot \sigma_{j+1}^Y \right\}. \quad (5)$$

We first rotate about the y -axis to give

$$H' = -J \sum_{j=1}^N \left\{ \sqrt{\left(\frac{2\lambda \text{Re}(\alpha)}{J\sqrt{N}} \right)^2 + \left(\frac{\epsilon}{2J} \right)^2} \sigma_j^{Z'} + \sigma_j^Y \cdot \sigma_{j+1}^Y \right\}. \quad (6)$$

We note here that the resulting hamiltonian is of the type of an Ising hamiltonian with a transverse field, and it exhibits a divergence in concurrence at its quantum phase transition (see, e.g., [7]). This particular model is instrumental in understanding the nature of coherence in quantum systems. Going back to the calculations, we may now

diagonalise H' by performing a Jordan-Wigner transformation, passing into momentum-space and then performing a Bogoliubov transformation (see, for example, Ref. [6]). We then have, in terms of momentum-space fermion operators γ_k , the diagonalised H' :

$$H' = \sum_{k=1}^N \xi_k(\alpha) (\gamma_k^\dagger \gamma_k - \frac{1}{2}) \quad (7)$$

with

$$\xi_k(\alpha) = 2J\sqrt{1 + (g(\alpha))^2 - 2g(\alpha)} \quad (8)$$

$$g(\alpha) = \sqrt{\left(\frac{2\lambda\text{Re}(\alpha)}{J\sqrt{N}}\right)^2 + \left(\frac{\epsilon}{2J}\right)^2}. \quad (9)$$

We may then write

$$H' = \sum_{k=1}^N H_k \quad (10)$$

where

$$H_k = \xi_k(\alpha) (\gamma_k^\dagger \gamma_k - \frac{1}{2}). \quad (11)$$

From the transformation, we may associate the spin-up state with an empty orbit on the site and a spin-down state with an occupied orbital. Using the commutation relations for the γ_k and the fact that $\gamma_k|0\rangle = 0$ (see, for example, Ref. [6]), we obtain

$$Z(N, T) = \int \frac{d^2\alpha}{\pi} e^{-\beta|\alpha|^2} \prod_{k=1}^N \{e^{-\frac{\beta}{2}\xi_k(\alpha)} + e^{\frac{\beta}{2}\xi_k(\alpha)}\}. \quad (12)$$

Writing $d^2\alpha = d\text{Re}(\alpha)d\text{Im}(\alpha)$, $w = \text{Re}(\alpha)$ and integrating out $\text{Im}(\alpha)$ we obtain

$$Z(N, T) = \frac{1}{\sqrt{\beta\pi}} \int dw e^{-\beta w^2 + \sum_{k=1}^N \{\log[\cosh(\frac{\beta}{2}\xi_k(x))] + \log(2)\}}. \quad (13)$$

We now let $x = w/\sqrt{N}$. Writing $\sum_{k=1}^N$ as $\frac{N}{2\pi} \int_0^{2\pi} dk$, yields

$$Z(N, T) = \sqrt{\frac{N}{\beta\pi}} \int_{-\infty}^{\infty} dx \left\{ e^{-\beta x^2 + I(x)} \right\}^N \quad (14)$$

where

$$I(x) = \frac{1}{2\pi} \int_0^{2\pi} dk \left\{ \log \left[\cosh \left(\frac{\beta}{2} \xi_k(x) \right) \right] + \log(2) \right\} \quad (15)$$

and

$$\xi_k(x) = 2J\sqrt{1 + (g(x))^2 - 2g(x) \cos k}. \quad (16)$$

From here on, we omit the $\log(2)$ term in $I(x)$ since it only contributes an overall factor to $Z(N, T)$.

Laplace's method now tells us that

$$Z(N, T) \propto \max_{-\infty \leq x \leq \infty} \exp \left\{ N[-\beta x^2 + I(x)] \right\}. \quad (17)$$

Denoting $[-\beta x^2 + I(x)]$ by $\Omega(x)$, we recall that the super-radiant phase corresponds to $\Omega(x)$ having its maximum at a non-zero x [11]. If there is no transverse field, i.e., if

$J = 0$, and the temperature is fixed, then the maximum of $\Omega(x)$ will split continuously into two maxima symmetric about the origin as λ^2 increases. Hence the process is a continuous phase transition.

However the case of non-zero J is qualitatively different from $J = 0$. As a result of the frustration induced by the transverse nearest-neighbour couplings, *there are regions where the super-radiant phase transition becomes first-order*. Hence *the system's phase transition can be driven to become first-order by suitable adjustment of the nearest-neighbour couplings*. This phenomenon of first-order phase transitions is revealed by considering the functional shape of $\Omega(x)$, as shown in Fig. 1.

Figure 2 shows the value of x that maximises $\Omega(x)$ at fixed ϵ and two different values of J . From the two lines, we can see that the spin-spin coupling actually acts to inhibit the phase transition. As we increase J from 0.8 to 1.0 we can see that the value to which we have to increase λ to induce a phase transition is higher.

Figure 3 plots the maximiser of $\Omega(x)$ with λ fixed at a value of 1.3. For small J , the local (non-zero) maximum of $\Omega(x)$ converges to zero as we increase ϵ and the system is no longer super-radiant. This is no longer the case if J is increased. In this case, $\Omega(x)$ has a global maximum when ϵ is small; however as ϵ increases, the non-zero local maxima becomes dominant and as a result a first-order phase transition occurs. We note that the barriers between the wells are infinite in the thermodynamic limit, hence we expect that the sub-radiant state is metastable as ϵ increases. This observation also suggests the phenomenon of hysteresis, which awaits experimental validation.

In Fig. 4 we consider the order parameter of the transition, $\langle \frac{a^\dagger a}{N} \rangle$. Following the same method as above, we may calculate this to be equivalent to x^2 with an additional $\frac{1}{2\beta}$ term that comes from the imaginary part of the coherent states of the radiation field [22]. We can see from the figure that as we lower β we drive the system first through a first order phase transition and then through a continuous phase transition. Thus we are able to achieve both a first and second order phase transition by varying the one parameter, β .

4. Conclusion

In conclusion, we have shown that the experimentally relevant spin-spin interaction in the Dicke model transforms it into an Ising-hamiltonian with a photon-field dependent transverse field, which allows for an existence of both first-order and second-order phase transitions as parameters vary. Our results highlight the importance of spin-spin coupling terms in spin-boson systems and opens up the possibility of coherently controlling the competition between the sub-radiant and super-radiant states in experimental atom-radiation systems [21].

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Figure 1. Demonstration as to what the function $\Omega(x)$ looks like across a first-order phase transition as λ and x are varied. Here $J = 1.0$, $\epsilon = 1.1$ and $\beta = 100.0$.

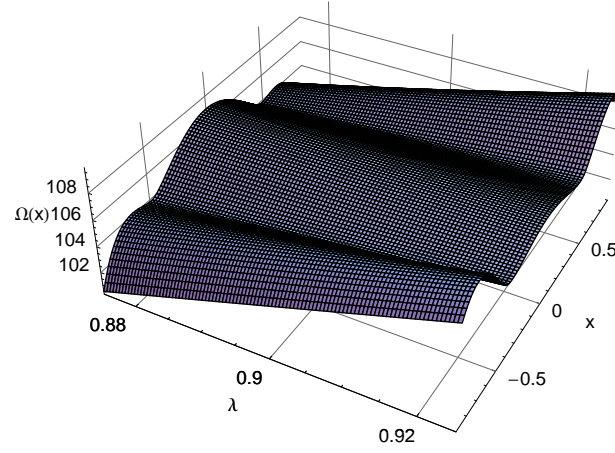


Figure 2. The value of x at which there is a maximum in $\Omega(x)$ as λ increases, for $J = 1.0$ (dashed line) and $J = 0.8$ (solid line). In both cases $\epsilon = 1.1$ and $\beta = 100$.

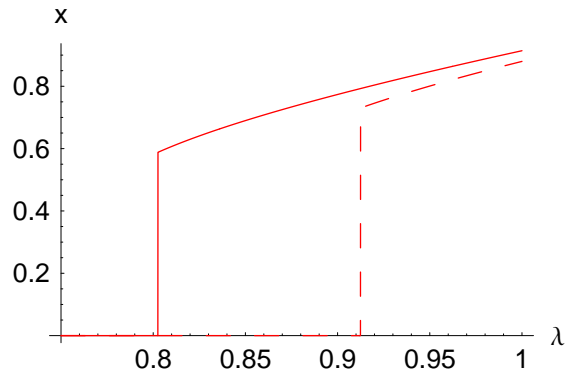


Figure 3. The maximiser of Ω shown as a function of J and ϵ . Here $\lambda = 1.3$ and $\beta = 100$.

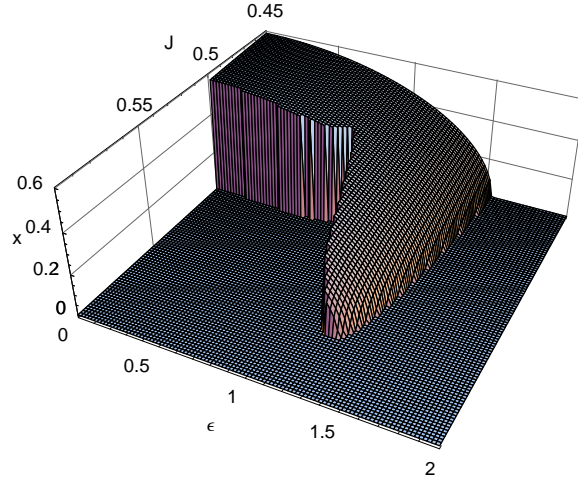


Figure 4. Plot of the order parameter, $\Theta = \langle \frac{a^\dagger a}{N} \rangle$, for the phase transition with $\lambda = 0.9$, $J = 1.0$ and $\epsilon = 1.1$.

